Journal of Pure and Applied Algebra 6 (1975) 133–153

THE BIVARIANT LONG EXACT SEQUENCE FOR THE EXT FUNCTOR

I.S. PRESSMAN
Carleton University, Ottawa, Ont., Canada

Communicated by Peter Hilton
Received September 11, 1974

0. Introduction

It is traditional to present the Ext functor in two distinct formats: as a covariant functor on the second variable with the first held fixed, and as a contravariant functor on the first variable with the second held fixed. In this paper, the situation where both variables are allowed to vary simultaneously is described, so that Ext acts in a *bivariant* manner. If one has two short exact sequences

$$\Phi : 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0, \qquad \Omega : 0 \to X \xrightarrow{\gamma} Y \xrightarrow{\delta} Z \to 0$$

one proves (Theorem 2.7) that there is a long exact sequence

$$S(\Omega, \Phi) : 0 \to \mathrm{Hom}(Z,A) \xrightarrow{\rho^0} \mathrm{Hom}(Y,B) \xrightarrow{\sigma^0} \mathrm{Hom}(\gamma,\beta)$$

$$\xrightarrow{\tau^0} \mathrm{Ext}^1(Z,A) \xrightarrow{\rho^1} \mathrm{Ext}^1(Y,B) \xrightarrow{\sigma^1} \mathrm{Ext}^1(\gamma,\beta) \xrightarrow{\tau^1} \ldots$$

which generalizes the usual two long exact sequences in that for proper choices of Φ or Ω one obtains either the contravariant or covariant long exact sequences together with a commutative diagram to the bivariant sequence from each of the others (see 3.1). The groups $\mathrm{Ext}^n(\gamma,\beta)$ are computed in the category of morphisms of an abelian category. S is a functor from the category of pairs of short exact sequences to the category of long exact sequences of abelian groups which is contravariant in the first and covariant in the second variable. The morphisms of S are completely described and are not difficult to use.

The elements of $\mathrm{Ext}^{n+1}(Z,A)$ which are images of τ^n, the connecting homomorphisms, are interpreted in 3.5 as obstructions to factoring commutative squares, and in 3.6 as obstructions to factoring diagrams of long exact sequences. τ^1 can be computed with particular ease.

Exact squares are studied in Section 4. A five term exact sequence is introduced which is exact if and only if the square is exact. A square is exact iff it is the middle square of an extension Γ of an epimorphism β by a monomorphism γ of length 2 (see 4.2). Then using the notation of the long exact sequence S, τ^2 of the Ext-class of Γ in $\mathrm{Ext}^3(Z,A)$ represents the obstruction to finding an exact square *congruent* to the given one, which can be factored (Theorem 4.7).

We then generalize the concept of exact square to that of an n-exact square, which is a commutative diagram $\mathfrak{X}^n$

$$\mathfrak{X}^n: \quad \begin{array}{ccccccc} X_1 & \to X_2 & \to \dots & \to X_{n-1} & \to X_n \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ Y_1 & \to Y_2 & \to \dots & \to Y_{n-1} & \to Y_n \end{array}$$

whose rows are exact; such that there is an epimorphism β with kernel A, and a monomorphism γ with cokernel Z such that the following diagram is commutative and has exact rows:

$$\Gamma^n: \quad \begin{array}{ccccccccc} 0 \to B & \to X_1 & \to X_2 & \to \dots & \to X_{n-1} & \to X_n & \to X \to 0 \\ \downarrow{\scriptstyle\beta} & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow{\scriptstyle\gamma} \\ 0 \to C & \to Y_1 & \to Y_2 & \to \dots & \to Y_{n-1} & \to Y_n & \to Y \to 0. \end{array}$$

Then there is a long exact sequence Δ^{n+1} which is called a generalized Mayer-Vietoris sequence:

$$\Delta^{n+1}: 0 \to A \to X_1 \to X_2 \oplus Y_1 \to X_3 \oplus Y_2 \to \dots \to X_n \oplus Y_{n-1} \to Y_n \to Z \to 0.$$

This sequence is interesting in itself. It also is true that Δ^{n+1} is the image of Γ^n under the connecting homomorphism τ^n. In the special case where every third morphism $X_{3k} \to Y_{3k}$ is the identity, one can map Δ^{n+1} to the usual Mayer–Vietoris sequence. This kind of argument can be carried over to more complex situations, such as when one has a commutative "square" of four long exact sequences.

This paper was presented at the 28th Summer Meeting of the Canadian Mathematical Congress. This research was supported by NRC grant No. A8505.

1. Notation

Let $\mathcal{A}$ denote a fixed abelian category, and $\mathcal{A}^2$ the category whose objects are all the morphisms of $\mathcal{A}$ and morphisms $\binom{\rho}{\sigma} \in \mathrm{Hom}(\gamma, \beta)$ correspond to commutative diagrams

$$(1.1) \qquad \begin{array}{ccc} \cdot & \xrightarrow{\rho} & \cdot \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ \cdot & \xrightarrow[\sigma]{} & \cdot \end{array} \qquad \beta\rho = \sigma\gamma.$$

The symbols 1_A, 0, $D0$, $0D$ will denote respectively the identity morphism $A \to A$, and the zero morphisms $0 \to 0$, $D \to 0$, and $0 \to D$.

In each abelian category there is a connected sequence of extension functors, $\mathrm{Ext}^n(-,-)$. If E

$$E : 0 \to B \to X_1 \to X_2 \to \ldots \to X_n \to A \to 0$$

is an exact sequence of length n in $\mathcal{A}$, we shall call it an n-fold extension of B by A; and we denote the class of E in $\mathrm{Ext}^n(A, B)$ by $[E]$. If $\varphi : B \to B'$ is a homomorphism, then we denote the n-fold extension of B' by A obtained by pushing out along φ by φE. Dually we obtain $E\mu$ by pulling back along μ. If F is an m-fold extension of C by B, then FE will denote the $(m+n)$fold extension of C by A obtained by splicing the two sequences together. Standard results about the extension functor can be found in Mitchell [5, Ch. 7]. In particular, note that all this can be applied to $\mathcal{A}^2$, since it is an abelian category too.

Let $(\beta, \chi_1, \chi_2, \ldots, \chi_n, \alpha) : E \to E'$ denote a *morphism of n-fold extensions*, that is, a commutative diagram of exact sequences

$$\begin{tikzcd}
E \arrow[d] & : 0 \arrow[r] & B \arrow[r] \arrow[d, "\beta"] & X_1 \arrow[r] \arrow[d, "\chi_1"] & X_2 \arrow[r] \arrow[d, "\chi_2"] & \ldots \arrow[r] & X_n \arrow[r] \arrow[d, "\chi_n"] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
E' & : 0 \arrow[r] & B' \arrow[r] & X'_1 \arrow[r] & X'_2 \arrow[r] & \ldots \arrow[r] & X'_n \arrow[r] & A' \arrow[r] & 0.
\end{tikzcd}$$

Let us call a morphism of the form "$(1, \chi_1, \chi_2, \ldots, \chi_{n-1}, \chi_n, 1)$" a *congruence*; "$(\beta, \chi_1, 1, \ldots, 1, 1)$" a *pushout*; and "$(1, 1, \ldots, 1, \chi_n, \alpha)$" a *pullback*.

Lemma 1.2. *Each morphism of n-fold extensions can be factored into a pushout, followed by a congruence, followed by a pullback*:

$$\begin{tikzcd}
E \arrow[r, "pushout"] \arrow[rrr, bend right, "given\ morphism"'] & \cdot \arrow[r, "congruence"] & \cdot \arrow[r, "pullback"] & E.
\end{tikzcd}$$

The proof simply consists of pushing out E along β, pulling back E' along α, and then using the universality of pushouts and pullbacks to fill in the rest of the diagram.

Remark 1.3. 1. If one sets $\mathcal{E}^n(\mathcal{A})$ to be the category of all n-fold extensions and morphisms thereof, pushouts are not necessarily epimorphisms, nor pullbacks monomorphisms. They are, however, respectively epic and monic in the homotopy category $\mathcal{E}^n(\mathcal{A})_h$ whose morphisms are homotopy classes of morphisms (of complexes) [4, 8].

2. In general it is not possible to factor a morphism of n-fold extensions into a pullback followed by a pushout as may be readily observed in the case $n = 1$ (see (3.6)).

There are three functors $J, J', J'' : \mathcal{A} \to \mathcal{A}^2$ given by: $J(A) = 0A$, $J(f) = \binom{0}{f}$; $J'(A) = A0$, $J'(f) = \binom{f}{0}$; $J''(A) = 1_A$, $J''(f) = \binom{f}{f}$. There is a natural isomorphism $j_n : \mathrm{Ext}^n(A,B) \xrightarrow{\cong} \mathrm{Ext}^n(0A,0B)$ for all A, B, and integers $n \geq 0$ which is induced by sending the long exact sequence (les) in $\mathcal{A}$

$$E : 0 \to B \to X_1 \to X_2 \to \ldots \to X_n \to A \to 0$$

to the les

$$0 \to 0B \to 0X_1 \to 0X_2 \to \ldots \to 0X_n \to 0A \to 0$$

in $\mathcal{A}^2$. This les can be mapped to any other les of length n running from $0B$ to $0A$ with E on the bottom – so the top lines can be essentially ignored. The sequence in $\mathcal{A}$ represents the zero class iff the sequence in $\mathcal{A}^2$ does so too. Similarly there are canonical isomorphisms j'_n and j''_n.

The notation $\mathrm{Ext}^n(A,B)$ and $\mathrm{Ext}^n(0A,0B)$ certainly represents two different extension functors in two different categories, but it will always be clear from the context and the variables which functor is being used.

2. Computations in $\mathcal{A}^2$

Proposition 2.1. *For all X and Y in $\mathcal{A}$, and all integers $n \geq 0$*

(i) $\mathrm{Ext}^n(X0, 1_Y) = 0$
(ii) $\mathrm{Ext}^n(1_X, 0Y) = 0$
(iii) $\mathrm{Ext}^n(0X, Y0) = 0$.

Proof. (i) $\mathrm{Ext}^0(X0, 1_Y) = \mathrm{Hom}(X0, 1_Y) = 0$ since there is just the zero morphism from $X0$ to 1_Y. Let

$$0 \to 1_Y \xrightarrow{\binom{f}{1}} g \to X0 \to 0$$

denote a representative of a class in $\mathrm{Ext}^1(X0, 1_Y)$. Then $\binom{g}{1}\binom{f}{1} = \binom{1}{1} : 1_Y \to 1_Y$ gives a splitting of the short exact sequence (ses). Since every ses splits, $\mathrm{Ext}^1(X0, 1_Y) = 0$.

Let us now proceed by induction on n. Assume that $\mathrm{Ext}^{n-1}(X0, 1_Y) = 0$ for all X and all Y. Choose a sequence of length n

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "1"] & U_1 \arrow[r] \arrow[d, "g_1"] & U_2 \arrow[r] \arrow[d, "g_2"] & \ldots \arrow[r] & U_{n-1} \arrow[r] \arrow[d, "g_{n-1}"] & U_n \arrow[r] \arrow[d, "g_n"] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] & V_1 \arrow[r] & V_2 \arrow[r] & \ldots \arrow[r] & V_{n-1} \arrow[r] & V_n \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Let $c' = \operatorname{coker}(1 \to g_1) : C_1 \to D_1$ and $c'' = \operatorname{coker}(g_1 \to g_2) : C_2 \to D_2$ and let P denote the pushout that makes the following diagram of ses's commute:

$$g_2''g_2' = g_2 \qquad \begin{tikzcd} 0 \arrow[r] & C_1 \arrow[r] \arrow[d, "c'"] & U_2 \arrow[r] \arrow[d, "g_2'"] & C_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & D_1 \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "g_2''"] & C_2 \arrow[r] \arrow[d, "c''"] & 0 \\ 0 \arrow[r] & D_1 \arrow[r] & V_2 \arrow[r] & D_2 \arrow[r] & 0. \end{tikzcd}$$

There is then a map of the above les to the les

$$0 \to 1_Y \to 1_{V_1} \to g_2'' \to g_3 \to g_4 \to \dots \to g_n \to X0 \to 0$$

so the two sequences are in the same Ext-class. But the new sequence is obtained by splicing the ses

$$0 \to 1_Y \to 1_{V_1} \to 1_{D_1} \to 0$$

to the les

$$0 \to 1_{D_1} \to g_2'' \to g_3 \to \dots \to X0 \to 0$$

which is of length $n-1$. By the inductive hypothesis this sequence represents the zero Ext-class, and therefore the splice (or Yoneda product) also gives the zero class. Thus $\operatorname{Ext}^n(X0, 1) = 0$.

(ii) This follows from (i) by duality.

(iii) There is only one ses representing any class of $\operatorname{Ext}^1(0X, Y0)$, namely

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "1"] \arrow[d] & Y \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & X \arrow[r, "1"] & X \arrow[r] & 0. \end{tikzcd}$$

Thus $\operatorname{Ext}^1(0X, Y0) = 0$. Given a les of length n

$$\begin{matrix} 0 \to & Y \to & U_1 \to & U_2 \to \dots \to & U_{n-1} \to & U_n \to & 0 \to 0 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 \to & 0 \to & V_1 \to & V_2 \to \dots \to & V_{n-1} \to & V_n \to & X \to 0 \end{matrix}$$

one can map it to the les

$$\begin{matrix} 0 \to & Y \to & U_1 \to & U_2 \to \dots \to & U_{n-1} \to & U_n \to & 0 \to 0 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 \to & 0 \to & 0 \longrightarrow & 0 \longrightarrow \dots \to & 0 \longrightarrow & X \xrightarrow{1} & X \to 0 \end{matrix}$$

and then map to it, the les

$$\begin{array}{ccccccccccccccccc}
0 & \to & Y & \xrightarrow{1} & Y & \to & 0 & \to & \cdots & \to & 0 & \to & 0 & \longrightarrow & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \longrightarrow & 0 & \to & 0 & \to & \cdots & \to & 0 & \to & X & \xrightarrow{1} & X & \to & 0.
\end{array}$$

This last les represents the 0-class. Therefore $\mathrm{Ext}^n(0X, Y0) = 0$. □

Consider next, for A and Z any two objects in $\mathscr{A}$, the following ses's in $\mathscr{A}^2$

(2.2) $\qquad T_A : 0 \to 0A \to 1_A \to A0 \to 0 \qquad T_Z : 0 \to 0Z \to 1_Z \to Z0 \to 0$

and use the Ext functor to construct from these the large commutative diagram part of which is reproduced here. For notational convenience the symbol $\cdots\cdots$ over a group will mean that the group is the zero group. The specific descriptions of the morphisms are omitted but they can be readily supplied, for example:

$$\binom{0A}{1_A}_* = \mathrm{Ext}^n\left(0Z, \binom{0A}{1_A}\right): \mathrm{Ext}^n(0Z,0A) \to \mathrm{Ext}^n(0Z, 1_A).$$

The connecting homomorphism $\mathrm{Ext}^n(0Z, 0A) \xrightarrow{\cong} \mathrm{Ext}^{n+1}(Z0,0A)$ is an isomorphism and will be denoted by $\eta_{Z,A}$; where $\eta_{Z,A}[G] = [GT_Z]$ (see Diagram 1).

$$\begin{tikzcd}
\mathrm{Ext}^{n-1}(Z0,A0) \arrow[r] \arrow[d] & \mathrm{Ext}^n(Z0,0A) \arrow[r] \arrow[d] & \overset{\cdots\cdots}{\mathrm{Ext}^n(Z0,1_A)} \arrow[r] \arrow[d] & \mathrm{Ext}^n(Z0,A0) \arrow[d, "\cong"] \\
\mathrm{Ext}^{n-1}(1_Z,A0) \arrow[r] \arrow[d] & \overset{\cdots\cdots}{\mathrm{Ext}^n(1_Z,0A)} \arrow[r] \arrow[d] & \mathrm{Ext}^n(1_Z,1_A) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{Ext}^n(1_Z,A0) \arrow[d] \\
\overset{\cdots\cdots}{\mathrm{Ext}^{n-1}(0Z,A0)} \arrow[r] \arrow[d] & \mathrm{Ext}^n(0Z,0A) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{Ext}^n(0Z,1_A) \arrow[r] \arrow[d] & \overset{\cdots\cdots}{\mathrm{Ext}^n(0Z,A0)} \arrow[d] \\
\mathrm{Ext}^n(Z0,A0) \arrow[r, "\cong"] & \mathrm{Ext}^{n+1}(Z0,0A) \arrow[r] & \overset{\cdots\cdots}{\mathrm{Ext}^{n+1}(Z0,1_A)} \arrow[r] & \mathrm{Ext}^{n+1}(Z0,A0)
\end{tikzcd}$$

Diagram 1.

Corollary 2.3. *For all A and all Z in $\mathscr{A}$, there are natural isomorphisms*

$$\mathrm{Ext}^n(Z,A) \cong \mathrm{Ext}^n(Z0,A0) \cong \mathrm{Ext}^{n+1}(Z0,0A) \cong \mathrm{Ext}^n(0Z,0A)$$

$$\cong \mathrm{Ext}^n(0Z,1_A) \cong \mathrm{Ext}^n(1_Z,1_A) \cong \mathrm{Ext}^n(1_Z,A0)$$

for $n \geqslant 0$.

Proof. One connects the isomorphism $j'_n : \mathrm{Ext}^n(Z,A) \to \mathrm{Ext}^n(Z0,A0)$ to the sequence of isomorphisms of the diagram.

Consider next the two ses's in $\mathcal{A}^2$

$$\text{(2.4)}\qquad \Xi:\quad \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & B \arrow[r, "1"] \arrow[d, "1"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

$$\Psi:\quad \begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\gamma"] \arrow[d, "\gamma"] & Y \arrow[r, "\delta"] \arrow[d, "1"] & Z \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y \arrow[r, "1"] & Y \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd}$$

Let

$$\text{(2.5)}\qquad \Phi : 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0, \qquad \Omega : 0 \to X \xrightarrow{\gamma} Y \xrightarrow{\delta} Z \to 0$$

denote respectively the bottom and top lines of the above ses's. [1]

From Ξ and Ψ, which can be rewritten respectively as $0 \to 0A \to 1_B \to \beta \to 0$ and $0 \to \gamma \to 1_Y \to Z0 \to 0$ one obtains the "usual" doubly infinite Hom-Ext diagram; part of which is given in Diagram 2.

$$\begin{tikzcd}[column sep=small] \mathrm{Hom}(1_Y,0A) \arrow[r] \arrow[d] & \mathrm{Hom}(1_Y,1_B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(1_Y,\beta) \arrow[r] \arrow[d] & \mathrm{Ext}^1(1_Y,0A) \arrow[r] \arrow[d] & \mathrm{Ext}^1(1_Y,1_B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}^1(1_Y,\beta) \arrow[d] \\ \mathrm{Hom}(\gamma,0A) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(\gamma,1_B) \arrow[r] \arrow[d] & \mathrm{Hom}(\gamma,\beta) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\gamma,0A) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}^1(\gamma,1_B) \arrow[r] \arrow[d] & \mathrm{Ext}^1(\gamma,\beta) \arrow[d] \\ \mathrm{Ext}^1(Z0,0A) \arrow[r] \arrow[d] & \mathrm{Ext}^1(Z0,1_B) \arrow[r] \arrow[d] & \mathrm{Ext}^1(Z0,\beta) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^2(Z0,0A) \arrow[r] \arrow[d] & \mathrm{Ext}^2(Z0,1_B) \arrow[r] \arrow[d] & \mathrm{Ext}^2(Z0,\beta) \arrow[d] \\ \mathrm{Ext}^1(1_Y,0A) \arrow[r] & \mathrm{Ext}^1(1_Y,1_B) \arrow[r] & \mathrm{Ext}^1(1_Y,\beta) \arrow[r] & \mathrm{Ext}^2(1_Y,0A) \arrow[r] & \mathrm{Ext}^2(1_Y,1_B) \arrow[r] & \mathrm{Ext}^2(1_Y,\beta) \end{tikzcd}$$

Diagram 2.

Lemma 2.6. *Given two ses's Φ and Ω in $\mathcal{A}$ one can obtain two ses's Ξ and Ψ in $\mathcal{A}^2$, as above, from which one obtains the following natural isomorphisms for all $n \geqslant 0$*

(a) $\mathrm{Ext}^n(\gamma,1_B) \cong \mathrm{Ext}^n(Y,B)$;

(b) $\mathrm{Ext}^n(\gamma,0A) \cong \mathrm{Ext}^n(Z,A)$.

[1] It will be useful to recall later that $\Xi \leftrightarrow \Phi$ and $\Psi \leftrightarrow \Omega$ are related as shown. Note that we can obtain Ξ and Ψ from Φ and Ω.

Proof. From 2.1 and 2.3 and Diagram 2 we have

$$\mathrm{Ext}^n(\gamma, 1_B) \underset{\cong}{\leftarrow} \mathrm{Ext}^n(1_Y, 1_B) \overset{j''}{\underset{\cong}{\leftarrow}} \mathrm{Ext}^n(Y,B),$$

$$\mathrm{Ext}^n(\gamma, 0A) \underset{\cong}{\rightarrow} \mathrm{Ext}^{n+1}(Z0, 0A) \overset{\eta}{\underset{\cong}{\leftarrow}} \mathrm{Ext}^n(0Z, 0A) \overset{j}{\underset{\cong}{\leftarrow}} \mathrm{Ext}^n(Z,A).$$

Theorem 2.7. *Given any two ses's Φ and Ω in $\mathscr{A}$ there is a les*

$$S(\Omega,\Phi) : 0 \to \mathrm{Hom}(Z,A) \xrightarrow{\rho^0} \mathrm{Hom}(Y,B) \xrightarrow{\sigma^0} \mathrm{Hom}(\gamma,\beta) \xrightarrow{\tau^0}$$

$$\mathrm{Ext}^1(Z,A) \xrightarrow{\rho^1} \mathrm{Ext}^1(Y,B) \xrightarrow{\sigma^1} \mathrm{Ext}^1(\gamma,\beta) \xrightarrow{\tau^1}$$

$$\mathrm{Ext}^2(Z,A) \to \ldots$$

where

$$\rho^n([E]) = [\alpha E \delta], \qquad \sigma^n([F]) = [\left(\begin{smallmatrix}1\\ \beta\end{smallmatrix}\right) J''(F) \left(\begin{smallmatrix}\gamma\\ 1\end{smallmatrix}\right)],$$

$$\tau^n([G]) = j^{-1} \eta_{Z,A}^{-1} [\Xi G \Psi].$$

S is, in fact, a functor from the category of pairs of ses's of $\mathscr{A}$ to the category of long exact sequences of abelian groups, which is contravariant on Ω and covariant on Φ.

Proof. The second row of Diagram 2 is actually

$$0 \to \mathrm{Hom}(\gamma, 0A) \to \mathrm{Hom}(\gamma, 1_B) \to \mathrm{Hom}(\gamma,\beta) \to \mathrm{Ext}^1(\gamma, 0A)$$

$$\to \mathrm{Ext}^1(\gamma, 1_B) \to \mathrm{Ext}^1(\gamma,\beta) \to \mathrm{Ext}^2(\gamma, 0A) \to \ldots$$

By Lemma 2.6 it is isomorphic term by term with the desired les. Let us check that the morphisms ρ, σ, τ of $S(\Omega, \Phi)$ are indeed the correct ones: let us begin with ρ,

(2.8)
$$\begin{tikzcd}
 & \mathrm{Ext}^n(\gamma,0A) \arrow[r, "\left(\begin{smallmatrix}0B\\ \alpha\end{smallmatrix}\right)_*"] \arrow[d, "k"', "\cong"] & \mathrm{Ext}^n(\gamma,1_B) \\
\mathrm{Ext}^n(0Z,0A) \arrow[ur, "\left(\begin{smallmatrix}X0\\ \delta\end{smallmatrix}\right)^*"] \arrow[r, "\cong", "\eta_{Z,A}"'] & \mathrm{Ext}^{n+1}(Z0,0A) & \mathrm{Ext}^n(1_Y,1_B) \arrow[u, "\cong", "\left(\begin{smallmatrix}\gamma\\ 1_Y\end{smallmatrix}\right)^*"'] \\
\mathrm{Ext}^n(Z,A) \arrow[u, "j", "\cong"'] & & \mathrm{Ext}^n(Y,B). \arrow[u, "\cong", "j''"']
\end{tikzcd}$$

If one refers back to Lemma 2.6 one can verify (by carefully following the relevant diagrams) that the composition

$$(j'')^{-1} \left(\begin{smallmatrix}\gamma\\ 1_Y\end{smallmatrix}\right)^{*-1} \left(\begin{smallmatrix}0B\\ \alpha\end{smallmatrix}\right)_* k^{-1} \eta_{Z,A}\, j$$

does in fact give the morphism called ρ_n; where k denotes the morphism induced by splicing the ses Ψ to an extension of $0A$ by γ of length n. Similarly $\eta_{Z,A}$ is induced by splicing the ses T_Z. In fact $[\binom{X0}{\delta}\Psi] = [T_Z]$. Therefore, for $[H] \in \mathrm{Ext}^n(0Z,0A)$

$$k\binom{X0}{\delta}^* [H] = [H\binom{X0}{\delta}\Psi] = [HT_Z] = \eta_{Z,A}[H]$$

so the triangle in (2.8) commutes and $\binom{X0}{\delta}^*$ is an isomorphism too. Take $[E] \in \mathrm{Ext}^n(Z,A)$ and notice that the bottom row of

$$\binom{0B}{\alpha}_* \binom{X0}{\delta}^* j([E]) = [\binom{0B}{\alpha}(JE)\binom{X0}{\delta}]$$

is simply the les $\alpha E\delta$, and this can be taken to be the bottom line of its image under $\binom{\gamma}{1_Y}^{*-1}$. However, within this new Ext-class, one can find an element whose top and bottom lines are the same, namely $\alpha E\delta$, and j''^{-1} of this element is $\alpha E\delta$. Thus

$$\rho^n([E]) = [\alpha E\delta].$$

Next, take $[F] \in \mathrm{Ext}^n(Y,B)$ and observe that

$$\binom{1_B}{\beta}_* \binom{\gamma}{1_Y}^* [J''(F)] = [\binom{1_B}{\beta}J''(F)\binom{\gamma}{1_Y}] = \sigma^n([F]).$$

Finally, take $[G] \in \mathrm{Ext}^n(\gamma,\beta)$. This goes to $[\Xi G] \in \mathrm{Ext}^{n+1}(\gamma,0A)$, and $[\Xi G\Psi] \in \mathrm{Ext}^{n+1}(Z0,0A)$ under k. Therefore

$$\tau^n([G]) = j^{-1}\eta_{Z,A}^{-1}[\Xi G\Psi].$$

(Another formulation of τ is to fatten up the lower line of ΞG so that the Ext-class is unchanged, but the morphism of the top line into the bottom is monic with cokernel an exact sequence in $\mathrm{Ext}^{n+1}(Z,A)$, which is a representative of $\tau^n([G])$!)

If there is a morphism of ses's $\zeta : \Omega \to \Omega'$ in $\mathcal{A}$, then there is a unique morphism induced in $\mathcal{A}^2$, $\zeta^\# : \Psi \to \Psi'$, which in turn induces a canonical morphism between the large Hom-Ext diagrams corresponding to each sequence. This gives all the right commutative diagrams to prove that $S(\Omega, \Phi)$ is a contravariant functor in the first variable. Similarly, S is covariant in the second variable.

2.9. Remarks on excision

It is a direct consequence of Proposition 2.1(i) that we have the following excision isomorphisms for all objects W and any ses Φ:

$\mathrm{Ext}^n(W0,A0) \cong \mathrm{Ext}^n(W0,\beta)$ and $\mathrm{Ext}^n(W0,\alpha) \cong \mathrm{Ext}^n(W0,0C)$. By duality $\mathrm{Ext}^n(\delta,0D) \cong \mathrm{Ext}^n(X0,0D)$ and $\mathrm{Ext}^n(0Z,0D) \cong \mathrm{Ext}^n(\gamma,0D)$. Thus $\binom{X0}{\delta}^*$ of (2.8) is an excision isomorphism [7, Prop. 2.7].

3. Applications

3.1. Couniversality of $S(\Omega,\Phi)$

There are two well-known Hom-Ext les's in $\mathcal{A}$ corresponding to the usual covariant Hom-functor "Hom(Z, -)", and contravariant Hom-functor "Hom(-,A)". Both of these long exact sequences map canonically to $S(\Omega,\Phi)$. Apply the functor $\mathrm{Ext}^n(0Z,\text{-})$ to the ses $0 \to A0 \to \beta \to 1_C \to 0$, and use Proposition 2.1(iii) and Corollary 2.3 to prove that

$$\mathrm{Ext}^n(0Z,\beta) \cong \mathrm{Ext}^n(0Z,1_C) \cong \mathrm{Ext}^n(Z,C). \tag{3.2}$$

By duality, one has that $\mathrm{Ext}^n(\gamma,A0) \cong \mathrm{Ext}^n(X,A)$. The first isomorphism simply amounts to writing down the lower line of the diagram representing an Ext-class; the second is dually the upper line.

Form the two commutative diagrams of ses's,

$$\begin{tikzcd}
\Phi' : \arrow[d] & 0 \arrow[r] & A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & 0 \\
\Phi : & 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega : \arrow[d] & 0 \arrow[r] & X \arrow[r, "\gamma"] \arrow[d] & Y \arrow[r, "\delta"] \arrow[d, "\delta"] & Z \arrow[r] \arrow[d, "1"] & 0 \\
\Omega' : & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r, "1"] & Z \arrow[r] & 0.
\end{tikzcd}$$

By the functoriality of S one has the following commutative diagram of long exact sequences:

$$\begin{tikzcd}
S(\Omega',\Phi) : \arrow[d] & 0 \arrow[r] & \mathrm{Hom}(Z,A) \arrow[r] \arrow[d, equal, "1"] & \mathrm{Hom}(Z,B) \arrow[r] \arrow[d] & \mathrm{Hom}(0Z,\beta) \arrow[r] \arrow[d] & \mathrm{Ext}^1(Z,A) \arrow[r] \arrow[d, equal, "1"] & \mathrm{Ext}^1(Z,B) \arrow[r] \arrow[d] & \ldots \\
S(\Omega,\Phi) : & 0 \arrow[r] & \mathrm{Hom}(Z,A) \arrow[r] & \mathrm{Hom}(Y,B) \arrow[r] & \mathrm{Hom}(\gamma,\beta) \arrow[r] & \mathrm{Ext}^1(Z,A) \arrow[r] & \mathrm{Ext}^1(Y,B) \arrow[r] & \ldots \\
S(\Omega,\Phi') : \arrow[u] & 0 \arrow[r] & \mathrm{Hom}(Z,A) \arrow[r] \arrow[u, equal, "1"] & \mathrm{Hom}(Y,A) \arrow[r] \arrow[u] & \mathrm{Hom}(\gamma,A0) \arrow[r] \arrow[u] & \mathrm{Ext}^1(Z,A) \arrow[r] \arrow[u, equal, "1"] & \mathrm{Ext}^1(Y,A) \arrow[r] \arrow[u] & \ldots
\end{tikzcd}$$

By using the isomorphisms obtained above we can rewrite this as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(Z,A) \arrow[r, "{}'\rho^0"] \arrow[d, "1"'] & \mathrm{Hom}(Z,B) \arrow[r, "{}'\sigma^0"] \arrow[d] & \mathrm{Hom}(Z,C) \arrow[r, "{}'\tau^0"] \arrow[d] & \mathrm{Ext}^1(Z,A) \arrow[r, "{}'\rho^1"] \arrow[d, "1"'] & \mathrm{Ext}^1(Z,B) \arrow[r, "{}'\sigma^1"] \arrow[d] & \mathrm{Ext}^1(Z,C) \arrow[r, "{}'\tau^1"] \arrow[d] & {} \\
0 \arrow[r] & \mathrm{Hom}(Z,A) \arrow[r] & \mathrm{Hom}(Y,B) \arrow[r] & \mathrm{Hom}(\gamma,\beta) \arrow[r] & \mathrm{Ext}^1(Z,A) \arrow[r] & \mathrm{Ext}^1(Y,B) \arrow[r] & \mathrm{Ext}^1(\gamma,\beta) \arrow[r] & {} \\
0 \arrow[r] & \mathrm{Hom}(Z,A) \arrow[r] \arrow[u, "1"] & \mathrm{Hom}(Y,A) \arrow[r] \arrow[u] & \mathrm{Hom}(X,A) \arrow[r] \arrow[u] & \mathrm{Ext}^1(Z,A) \arrow[r] \arrow[u, "1"] & \mathrm{Ext}^1(Y,A) \arrow[r] \arrow[u] & \mathrm{Ext}^1(X,A) \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

Thus the middle les $S(\Omega,\Phi)$ has an unusual co-universal type property in that the usual Hom-Ext sequences both map into it. One should verify that the morphisms of the first and third rows are the usual one. We need only do the first, since the other case is dual.

3.3. Verification

(a) $'\rho^n([E]) = [\alpha E\ 1] = [\alpha E] = \alpha_*([E]) = \mathrm{Ext}^n(Z,\alpha)([E])$ so $'\rho^n$ is in fact the correct homomorphism

(b) When an exact sequence is subjected to several pushouts and pullbacks, its class is independent of the order of these operations [5, Lemma VII 1.3]. By Theorem 2.7

$$\begin{aligned}
'\sigma^n([F]) &= j''^{-1} \tbinom{0Z}{1_Z}^{*-1} \tbinom{\beta}{1}_* [\tbinom{1}{\beta} J''(F) \tbinom{0Z}{1}] \\
&= j''^{-1} \tbinom{0Z}{1_Z}^{*-1} [\tbinom{\beta}{1} \tbinom{1}{\beta} J''(F) \tbinom{0Z}{1}] \\
&= j''^{-1} \tbinom{0Z}{1_Z}^{*-1} \tbinom{0Z}{1_Z}^{*} [\tbinom{\beta}{\beta} J''(F)] \\
&= j''^{-1} [J''(\beta F)] = [\beta F] = \beta_*([F]).
\end{aligned}$$

Again this is the correct homomorphism.

(c) Note that in the process of obtaining Φ from Ω, one obtains T_Z from Ω', where T_Z is given in 2.2. Form the ses

$$\begin{array}{ccccccccc}
0 & \to & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle 1} & & \\
0 & \to & 0 & \longrightarrow & C & \xrightarrow{1} & C & \to & 0
\end{array}$$

and by 2.1 (iii) there is an isomorphism

$$\tbinom{\beta}{1_C}_* : \mathrm{Ext}^n(0Z,\beta) \xrightarrow{\cong} \mathrm{Ext}^n(0Z,1_C).$$

This is the first isomorphism in (3.2). The second isomorphism in (3.2) is given by

$$\mathrm{Ext}^n(Z,C) \underset{\cong}{\xrightarrow{j}} \mathrm{Ext}^n(0Z,0C) \underset{\cong}{\xrightarrow{\tbinom{0C}{1_C}_*}} \mathrm{Ext}^n(0Z,1_C).$$

Now

$$\tbinom{\beta}{1_C}\tbinom{0B}{1_C} = \tbinom{0C}{1_C}$$

so by the functoriality of $\mathrm{Ext}^n(0Z,\text{-})$

$$\tbinom{0B}{1_C}_* = \tbinom{\beta}{1_C}_*^{-1} \tbinom{0C}{1_C}_*$$

is an isomorphism. Next observe that for $[G] \in \mathrm{Ext}^n(Z,C)$

$$[\Xi\tbinom{0B}{1} J(G)] = [J(\Phi G)].$$

Thus

$$\begin{aligned}
'\tau^n([G]) &= j^{-1} \eta_{Z,A}^{-1} [\Xi\tbinom{0B}{1_C} J(G)\, T_Z] = j^{-1} \eta_{Z,A}^{-1} \eta_{Z,A} [\Xi\tbinom{0B}{1} J(G)] \\
&= j^{-1} [J(\Phi G)] = [\Phi G]
\end{aligned}$$

which is the familiar connecting homomorphism.

The verification of the other les is simply dual to all this.

3.4. Remark. The usual two Hom-Ext sequences can now be considered as special cases of $S(\Omega,\Phi)$.

3.5. Obstruction theory of commutative squares

Consider the next commutative diagram as a morphism $\binom{\varphi}{\varphi'}: \gamma \to \beta$, where γ is a monomorphism and β is an epimorphism as in (2.5),

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ Y \arrow[r, "\varphi'"'] \arrow[ur, dashed, "\lambda"] & C. \end{tikzcd}$$

From the exactness of $S(\Omega,\Phi)$ it follows that $\tau^0\binom{\varphi}{\varphi'} = 0 \in \operatorname{Ext}^1(Z,A)$ precisely when there is an element $\lambda \in \operatorname{Hom}(Y,B)$, such that

$$\sigma^0(\lambda) = \binom{1}{\beta} J''(\lambda)\binom{\gamma}{1} = \binom{1}{\beta}\binom{\lambda}{\lambda}\binom{\gamma}{1} = \binom{\lambda\gamma}{\beta\lambda} = \binom{\varphi}{\varphi'}.$$

Thus $\varphi = \lambda\gamma$ and $\varphi' = \beta\lambda$, and λ factors the square into two commutative triangles. Call such a λ a *lifting.*

Thus the conclusion is that if we have any commutative square $\varphi'\gamma = \beta\varphi$, with γ a monomorphism and β an epimorphism, then there is a lifting λ if and only if $\tau^0\binom{\varphi}{\varphi'}$ is the zero element of $\operatorname{Ext}^1(Z,A)$. In other words $\tau^0\binom{\varphi}{\varphi'}$ is the obstruction to finding such a factorization of commutative squares of this type. This was discussed in [6], but in a less satisfying manner. The obstruction $[G]$ of [6] is precisely $\tau^0\binom{\varphi}{\varphi'}$, the verification of which is not long.

Let us next entertain the case of 2 ses's

$$E' : 0 \to A \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C \to 0,$$

and

$$E'' : 0 \to A \xrightarrow{\alpha''} B \xrightarrow{\beta''} C \to 0$$

which form the commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha'"] \arrow[d, "\alpha''"'] & B' \arrow[d, "\beta'"] \\ B'' \arrow[r, "\beta''"'] & C. \end{tikzcd}$$

Then using the exact sequence $S(E'', E')$ we find that the morphism $\binom{\alpha'}{\beta''} \in \operatorname{Hom}(\alpha'',\beta')$ maps to $[E''] - [E']$ in $\operatorname{Ext}^1(C,A)$ under τ^0. Thus we have that $[E''] = [E']$ if and only if there is a lifting $B'' \to B'$ of the square.

3.6. Obstructions to factoring morphisms of n-fold extensions

In (1.2) we discussed the factorization of a morphism of an n-fold extension into a pushout, followed by a congruence, followed by a pullback. The impossibility of factoring every such morphism into a pullback followed by a pushout can be seen in the commutative diagram of ses's

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}_4 \arrow[r, "i"] \arrow[d, "0"] & \mathbf{Z}_{16} \arrow[r, "p"] \arrow[d, "p"] & \mathbf{Z}_4 \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & \mathbf{Z}_2 \arrow[r, "j"] & \mathbf{Z}_4 \arrow[r, "q"] & \mathbf{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

where i and j are the obvious inclusions of subgroups, and p and q the resulting cokernels. If this diagram were factored, it would have the following form:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}_4 \arrow[r] \arrow[d, equal] & \mathbf{Z}_{16} \arrow[r] \arrow[d] & \mathbf{Z}_4 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{Z}_4 \arrow[r] \arrow[d, "0"] & ? \arrow[r] \arrow[d] & \mathbf{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{Z}_2 \arrow[r] & \mathbf{Z}_4 \arrow[r] & \mathbf{Z}_2 \arrow[r] & 0.
\end{tikzcd}$$

But since the lower left hand square is a pushout the middle term of the bottom line must have a $\mathbf{Z}_2$ component since the morphism $\mathbf{Z}_4 \to \mathbf{Z}_2$ is the zero morphism! This is impossible.

However, let $[\mathrm{F}] \in \operatorname{Ext}^n(Y,B)$ be an extension class such that

$$\sigma^n([F]) = [\binom{1}{\beta} J''(F) \binom{\gamma}{1}] = [G].$$

If I denotes the identity morphism, then there is a commutative diagram of exact sequences,

$$\begin{tikzcd}
F \arrow[r, "I"] \arrow[d, "I"] & F \arrow[d, "{(\beta,\beta',1,\dots,1)}"] & F_1 \arrow[l, "{(1,\dots,1,\gamma',\gamma)}"'] \arrow[d, "{(\beta,\beta',1,\dots,1,\gamma',\gamma)}"] \\
F \arrow[r, "{(\beta,\beta',1,\dots,1)}"] & F_2 & F_2. \arrow[l, "I"']
\end{tikzcd}$$

But $(\beta,\beta',1, \dots, 1,\gamma',\gamma) : F_1 \to F_2$ represents an element $G^{\#}$ which is in the same extension class as G; and $G^{\#}$ can be factored as a pullback $F_1 \to F$ followed by a pushout $F \to F_2$.

Thus, given β an epimorphism, γ a monomorphism, and $G = (\beta, \chi_1, \chi_2, \dots, \chi_n, \gamma) : F_1' \to F_2'$, then $\tau^n([G]) \in \operatorname{Ext}^{n+1}(Z,A)$ represents the obstruction to finding an extension $G^{\#}$ of length n of β by γ which is in the same extension class as G, i.e. $[G] = [G^{\#}]$, and such that $G^{\#}$ can be factored into a pushout and then a pullback. We cannot make this remark about G itself in general, unless $n = 1$, where the middle morphism in each Ext^1-class is unique up to isomorphism.

3.7. An alternate means of computing τ^1

Let $G : 0 \to \beta \to \chi \to \gamma \to 0$ represent a class in $\mathrm{Ext}^1(\gamma,\beta)$. Since γ is a monomorphism $\mathrm{Ker}\,\chi \cong \mathrm{Ker}\,\beta = A$, and since β is an epimorphism $\mathrm{Coker}\,\chi \cong \mathrm{Coker}\,\gamma = Z$. This gives rise to an extension H of length 2 representing a class $[H] \in \mathrm{Ext}^2(Z,A)$,

$$\begin{tikzcd}
& 0 \arrow[r] & A \arrow[r] \arrow[d, "1"] & B \arrow[r, "\beta"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[d] \\
H: & 0 \arrow[r] & A \arrow[r] \arrow[d] & U \arrow[r, "\chi"] \arrow[d] & V \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "1"] & 0 \\
& & 0 \arrow[r] & X \arrow[r, "\gamma"] & Y \arrow[r] & Z \arrow[r] & 0.
\end{tikzcd}$$

Proposition 3.8. $\tau^1([G]) = [H]$.

Proof. $\Xi G\Psi$ is the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & B \arrow[r, "f_1"] \arrow[d, "1"] & U \arrow[r, "f_2"] \arrow[d, "\chi"] & Y \arrow[r] \arrow[d, "1"] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "g_1"] & V \arrow[r, "g_2"] & Y \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

There is a morphism from $\Xi G\Psi$ to the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & U \arrow[r] & V \arrow[r] & Z \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Therefore they represent the same class in $\mathrm{Ext}^3(Z0,0A)$. But the latter is just $J(H)T_Z$. Therefore

$$\tau^1([G]) = j^{-1}\,\eta_{Z,A}^{-1}\,[\Xi G\Psi] = j^{-1}\,\eta^{-1}\,\eta[J(H)] = [H].$$

3.9. Special case

Next, let us consider the special case where $X = C$. That is, keep Φ as above, but set

$$\Omega^* : 0 \to C \xrightarrow{\gamma^*} Y^* \xrightarrow{\delta^*} Z^* \to 0.$$

One can splice these two sequences to get $[\Phi\Omega^*] \in \mathrm{Ext}^2(Z^*,A)$. But

$$\rho^2([\Phi\Omega^*]) = [\alpha\Phi\Omega^*\delta^*] = 0$$

since $\alpha\Phi$ splits. By exactness there is a ses G, $[G] \in \mathrm{Ext}^2(\gamma^*,\beta)$, such that $\tau^1([G]) = [\Phi\Omega^*]$.

One can construct G quite easily. Let

$$\nu = \{\langle\beta, -1_C\rangle, \langle 0, \gamma^*\rangle\} : B \oplus C \to C \oplus Y^*$$

and form the ses G in $\mathscr{A}^2$,

$$G:\quad \begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\{1{,}0\}"] \arrow[d, "\beta"] & B \oplus C \arrow[r, "\langle 0{,}1\rangle"] \arrow[d, "\nu"] & C \arrow[r] \arrow[d, "\gamma^*"] & 0 \\ 0 \arrow[r] & C \arrow[r, "\{1{,}0\}"] & C \oplus Y^* \arrow[r, "\langle 0{,}1\rangle"] & Y^* \arrow[r] & 0. \end{tikzcd}$$

By (3.8), $\tau^1([G])$ is given by the exact sequence

$$H : 0 \to A \to B \oplus C \xrightarrow{\nu} C \oplus Y^* \to Z^* \to 0.$$

There is however a commutative diagram of exact sequences

$$\begin{tikzcd} \Phi\Omega^* : 0 \arrow[r] \arrow[d] & A \arrow[r, "\alpha"] \arrow[d, equal] & B \arrow[r, "\gamma^*\beta"] \arrow[d, "\langle 1{,}\beta\rangle"] & Y^* \arrow[r, "\delta^*"] \arrow[d, "\langle 0{,}1\rangle"] & Z^* \arrow[r] \arrow[d, equal] & 0 \\ H\ : 0 \arrow[r] & A \arrow[r, "\langle\alpha{,}0\rangle"] & B \oplus C \arrow[r, "\nu"] & C \oplus Y^* \arrow[r, "\langle 0{,}\delta^*\rangle"] & Z^* \arrow[r] & 0. \end{tikzcd}$$

Thus $\tau^1([G]) = [H] = [\Phi\Omega^*]$, so in this special case one can explicitly describe an inverse image of $\Phi\Omega^*$ under $(\tau^1)^{-1}$.

4. Exact squares

$$\text{(4.1)} \qquad \begin{tikzcd} K \arrow[r, "\lambda"] \arrow[d, "\pi"'] & L \arrow[d, "\nu"] \\ M \arrow[r, "\mu"] & N \end{tikzcd}$$

Consider the commutative diagram (4.1), and let

$$\beta = \mathrm{Ker}\binom{\lambda}{\mu} : B \to C, \qquad \gamma = \mathrm{Coker}\binom{\lambda}{\mu} : X \to Y.$$

Construct the exact sequence Γ of length two

$$\Gamma:\quad \begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\pi"] \arrow[d, "\beta"] & K \arrow[r, "\lambda"] \arrow[d, "\kappa"] & L \arrow[r, "\omega"] \arrow[d, "\nu"] & X \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & C \arrow[r, "\xi"] & M \arrow[r, "\mu"] & N \arrow[r, "\upsilon"] & Y \arrow[r] & 0. \end{tikzcd}$$

Definition 4.2. (4.1) is an *exact square* iff β is an epimorphism and γ is a monomorphism.

We shall continue with the notation of (2.5) so that $\alpha = \ker \beta$ and $\delta = \operatorname{coker} \gamma$.

Theorem 4.3. [Brinkmann and Puppe]. *The following are equivalent:*

(a) (4.1) *is exact;*

(b) *if λ (respectively μ) are factored into an epimorphism λ'' (respectively μ'') followed by a monomorphism λ' (μ'), and ζ is the induced morphism*

$$\begin{tikzcd} K \arrow[r, "\lambda''"] \arrow[d, "\kappa"'] & I \arrow[r, "\lambda'"] \arrow[d, "\zeta"] & L \arrow[d, "\nu"] \\ M \arrow[r, "\mu''"] & J \arrow[r, "\mu'"] & N \end{tikzcd}$$

then the left square is a pushout, and the right a pullback,

(c) $K \xrightarrow{\{\lambda,\kappa\}} L \oplus M \xrightarrow{\langle -\nu,\mu\rangle} N$ *is exact.*

The proof is found in [1, p. 42–44]. Let us construct from Γ the sequence Δ

$$\Delta : 0 \to A \xrightarrow{\pi\alpha} K \xrightarrow{\{\lambda,\kappa\}} L \oplus M \xrightarrow{\langle -\nu,\mu\rangle} N \xrightarrow{\delta\upsilon} Z \to 0.$$

Theorem 4.4. (4.1) *is an exact square iff Δ is an exact sequence.*

Proof. If Δ is exact, then condition (c) of 4.3 is satisfied so (4.1) is exact. Conversely if (4.1) is exact, we have exactness at $L \oplus M$. The sequence is exact at A since $\pi\alpha$ is monic. To show exactness at K note that

$$\{\lambda,\kappa\}\pi\alpha = \{\lambda\pi\alpha,\kappa\pi\alpha\} = \{0, \xi\beta\alpha\} = \{0,0\} = 0.$$

Conversely if $\{\lambda,\kappa\}\psi = 0$ for some morphism ψ, then $\lambda\psi = 0$ so $\psi = \pi\psi'$; and $\kappa\psi = 0$, so $\kappa\psi = \kappa\pi\psi' = \xi\beta\psi' = 0$. But ξ is monic so $\beta\psi' = 0$ so $\psi' = \alpha\psi''$. Thus $\psi = \pi\alpha\psi''$, so ψ factors through $\pi\alpha$. Therefore the sequence is exact at K. Dually it is exact at N and Z.

Suppose now that (4.1) is exact and that there is a second exact square

$$(4.1)^* \qquad \begin{tikzcd} K^* \arrow[r, "\lambda^*"] \arrow[d, "\kappa^*"'] & L^* \arrow[d, "\nu^*"] \\ M^* \arrow[r, "\mu^*"] & N^* \end{tikzcd}$$

and morphisms $K \to K^*$, $L \to L^*$, $M \to M^*$, $N \to N^*$ giving a commutative diagram such that the induced morphisms

$$\operatorname{Ker}\binom{\lambda}{\mu} \to \operatorname{Ker}\binom{\lambda^*}{\mu^*}, \qquad \operatorname{Coker}\binom{\lambda}{\mu} \to \operatorname{Coker}\binom{\lambda^*}{\mu^*}$$

are both the identity. Call such a diagram a congruence of exact squares because it induces a congruence (see 1.2) of extensions of β by γ. Two squares connected by a finite sequence of congruences in either direction are called congruent. Suppose (4.1)* gives rise to the exact sequence Γ^*. Then there is a congruence (4.1) $\to$ (4.1)* iff there is a congruence $\Gamma \to \Gamma^*$.

Let us now recall the connecting homomorphism τ^2 of Theorem 2.7. Suppose that (4.1) is exact.

Lemma 4.5. $\tau^2([\Gamma]) = [\Delta]$.

Proof. $\tau^2([\Gamma]) = j^{-1}\eta^{-1}[\Xi\Gamma\psi]$ where $\Xi\Gamma\psi$ is the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & B \arrow[r, "\pi"] \arrow[d, "1"] & K \arrow[r, "\lambda"] \arrow[d, "\kappa"] & L \arrow[r] \arrow[d, "\nu"] & Y \arrow[r, "\delta"] \arrow[d, "1"] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r] & M \arrow[r, "\mu"] & N \arrow[r, "\nu"] & Y \arrow[r] & 0 \arrow[r] & 0 .
\end{tikzcd}$$

There is a morphism from this exact sequence to the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & K \arrow[r, "\{0,1\}"] \arrow[d, "1"] & L \oplus K \arrow[r, "\langle 1,0\rangle"] \arrow[d, "\epsilon"] & L \arrow[r, "0"] \arrow[d, "\nu"] & Z \arrow[r, "1"] \arrow[d, "1"] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\pi\alpha"] & K \arrow[r, "\{\lambda,\kappa\}"] & L \oplus M \arrow[r, "\langle -\nu,\mu\rangle"] & N \arrow[r, "\delta\nu"] & Z \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where $\epsilon = \langle\{-1,0\},\ \{\lambda,\kappa\}\rangle$, and the morphisms between the exact sequences are the identity except for $\pi : B \to K$, $\{\lambda,1\} : K \to L \oplus K$, $\delta : Y \to Z$, and $\{0,1\} : M \to L \oplus M$. There is obviously a similar exact sequence S', with the $K \to L \oplus K \to L$ part on top deleted, which maps onto this last sequence, and $j^{-1}\eta^{-1}(S') = \Delta$.

Lemma 4.6. *The exact square* (4.1) *has a lifting* $\ell : M \to L$ *iff* $\zeta = \operatorname{Im}\binom{\lambda}{\mu}$ *is an isomorphism* (*see Theorem* 4.3(b)).

Proof. If ζ is an isomorphism then $\ell = \lambda'\zeta^{-1}\mu''$ is a lifting of (4.1). Conversely, if there is a lifting ℓ, then by [6, Theorem 2] the induced morphisms Ker $\kappa \to$ Ker ν and Coker $\kappa \to$ Coker ν are both zero. Therefore Ker $\zeta = 0$, Coker $\zeta = 0$, so ζ is an isomorphism.

Remark. If (4.1) has a lifting then because of the exactness of (4.1) it follows that κ is an epimorphism and ν a monomorphism.

Theorem 4.7. *Let* Γ *be the exact sequence constructed from the exact square* (4.1). *The exact square is congruent to an exact square with a lifting iff* $\tau^2([\Gamma]) = 0$. *That is,* $\tau^2([\Gamma])$ *is the obstruction to finding a congruent square with a lifting.*

Proof. First let us assume that (4.1) has a lifting. Since ζ is an isomorphism, $\zeta\lambda''$ is a cokernel of $\pi : B \to K$ so there is an exact sequence G

$$G : 0 \to B \xrightarrow{\pi} K \xrightarrow{\mu\kappa} N \xrightarrow{\upsilon} Y \to 0.$$

It is clear that $\sigma^2([G]) = [\Gamma]$, so $\tau^2\sigma^2([G]) = 0$. A similar outcome would ensue if it were a square congruent to (4.1), and not (4.1) itself, which had the lifting.

Conversely if $\tau^2([\Gamma]) = 0$, then by the exactness of the les of Theorem 2.4, there is an exact sequence G''

$$G'' : 0 \to B \to K'' \to N'' \to Y \to 0$$

such that $\sigma^2([G'']) = [\Gamma]$. But $\binom{1}{\beta} J''(G'') \binom{\gamma}{1}$ is an extension of β by γ of length 2, and its middle square has the form

$$(4.1)'' \qquad \begin{array}{ccc} K'' & \xrightarrow{\lambda''} & L'' \\ \downarrow & & \downarrow \\ M'' & \xrightarrow{\mu''} & N'' \end{array}$$

and is exact. Moreover, by the construction $\zeta'' = \operatorname{Im}\binom{\lambda''}{\mu''}$ can be taken to be the identity on $\operatorname{Im}(K'' \to L'')$. Thus by Lemma 4.6, the exact square (4.1)″ has a lifting. Clearly (4.1)″ is congruent to (4.1).

One should mention that these liftings are dissimilar from those of 3.5. In the earlier case, using the notation of (4.7), one would need κ monic and ν epic, and then we are asking for a lifting of the particular square. In this case, we only ask for a lifting of a congruent square, and in the event that one does exist one finds that κ is epic and ν monic. The obstructions to the first lifting are in $\operatorname{Ext}^1(Z,A)$, and to the second in $\operatorname{Ext}^3(Z,A)$.[2]

Definition 4.8. Suppose that one has a commutative diagram $\mathfrak{X}^n$

$$\mathfrak{X}^n : \quad \begin{array}{ccccccccc} X_1 & \xrightarrow{\delta_1} & X_2 & \xrightarrow{\delta_2} & \cdots \to & X_{n-1} & \xrightarrow{\delta_{n-1}} & X_n \\ \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_2} & & & \downarrow{\scriptstyle \chi_{n-1}} & & \downarrow{\scriptstyle \chi_n} \\ Y_1 & \xrightarrow{\delta_1'} & Y_2 & \xrightarrow{\delta_2'} & \cdots \to & Y_{n-1} & \xrightarrow{\delta_{n-1}'} & Y_n \end{array}$$

with exact rows, such that

$$\Gamma^n : 0 \to \beta \xrightarrow{\binom{\pi}{\xi}} \chi_1 \to \chi_2 \to \cdots \to \chi_n \xrightarrow{\binom{\omega}{\upsilon}} \gamma \to 0$$

is an exact sequence, with β an epimorphism and γ a monomorphism. Let such a diagram be called an *n-exact square.*

It is clear that an exact square is a 2-exact square. It turns out to be possible to generalize the results for 2-exact squares to n-exact squares.

[2] If homological dimension of $\mathscr{A}$ is $\leqslant 2$, the obstructions vanish!

Let us assume that β and γ are our usual morphisms (2.5). Consider Δ^{n+1}, which has the mapping cone for $\mathfrak{X}^n$ between X_1 and Y_n,

$$\Delta^{n+1}: 0 \to A \xrightarrow{\pi\alpha} X_1 \to X_2 \oplus Y_1 \to X_3 \oplus Y_2 \to X_4 \oplus Y_3 \to \ldots \to X_n \oplus Y_{n-1} \to Y_n \xrightarrow{\delta v} Z \to 0$$

where the morphisms are

$$\{\delta_1, -\chi_1\} : X_1 \to X_2 \oplus Y_1, \qquad \langle (-1)^n \chi_n, \delta'_{n-1} \rangle : X_n \oplus Y_{n-1} \to Y_n,$$

and

$$\langle \{\delta_i, (-1)^i \chi_i\}, \{0, \delta'_i\} \rangle : X_i \oplus Y_{i-1} \to X_{i+1} \oplus Y_i \quad \text{for } 2 \leqslant i \leqslant n-1.$$

Even if β is not an epimorphism and γ is not a monomorphism, Δ^{n+1} is exact at $X_i \oplus Y_{i-1}$ for $3 \leqslant i \leqslant n-2$ if Γ^n is exact.

Theorem 4.9. *$\mathfrak{X}^n$ is an n-exact square iff Δ^{n+1} is exact.*

Lemma 4.10. $\tau^n[\Gamma^n] = [\Delta^{n+1}]$.

Theorem 4.11. $\tau^n[\Gamma^n] = 0 \in \mathrm{Ext}^{n+1}(Z,A)$ *iff Γ^n is congruent to an extension of β by γ which can be factored into a pullback followed by a pushout* (3.6).

The proofs of these are straightforward. It should be pointed out that factorizability into a pullback followed by a pushout is indeed the generalization of the concept of the existence of a lifting in an exact square.

Consider the special case $\mathfrak{X}^{3k+1}$ where $\chi_1, \chi_4, \ldots, \chi_{3k+1}$ are all identity morphisms. Observe that $\gamma_1 = \mathrm{Im}(\begin{smallmatrix}\delta_1 \\ \delta'_1\end{smallmatrix})$ is an epimorphism, and $\beta_{3k} = \mathrm{Im}(\begin{smallmatrix}\delta_{3k} \\ \delta'_{3k}\end{smallmatrix})$ is a monomorphism so one obtains an exact sequence similar to Δ^{n+1}

$$X_2 \to X_3 \oplus Y_2 \to X_4 \oplus Y_3 \to X_5 \oplus Y_4 \to \ldots \to X_{3k} \oplus Y_{3k-1} \to Y_{3k}. \tag{4.12}$$

But $X_4 = Y_4$, $X_1 = Y_1$, etc. so one may delete them in the new exact sequence

$$\mathcal{M}: X_2 \to X_3 \oplus Y_2 \to Y_3 \xrightarrow{-\delta_4\delta'_3} X_5 \to X_6 \oplus Y_5 \to Y_6 \xrightarrow{\delta_7\delta'_6} X_8 \to \ldots \to Y_{3k}.$$

There is, in fact, an epimorphism of the first sequence (4.12) onto $\mathcal{M}$ which is the identity on the ends and on the $X_{3m} \oplus Y_{3m-1}$ terms, and has the form

$$\langle 0,1 \rangle : X_4 \oplus Y_3 \to Y_3, \qquad \langle 1, -\delta_4 \rangle : X_5 \oplus Y_4 \to X_5 \text{ (i.e. } Y_4 = X_4),$$

and generally

$$\langle 1, (-1)^{3m+1} \delta_{3m+1} \rangle : X_{3m+2} \oplus Y_{3m+1} \to X_{3m+2}.$$

The exact sequence $\mathcal{M}$ is the Mayer–Vietoris sequence. Thus it would be reasonable to call Δ^{n+1} a *generalized Mayer–Vietoris* sequence.

Another interesting case occurs when $\chi_1, \chi_3, \chi_5, \ldots, \chi_{2m+1}, \ldots, \chi_{2k+1}$ are all isomorphisms. One then obtains, in a manner analogous to the above, the exact sequence

$$X_2 \to Y_2 \to X_4 \to Y_4 \to X_6 \to \ldots \to X_{2k} \to Y_{2k}.$$

One occasionally encounters the situation where one has four exact sequences forming a commutative square of exact sequences. If one should treat this as a morphism from $\Gamma^n \to \Gamma^{n''}$, where

$$\Gamma^{n''}: 0 \to \beta'' \to \chi_1'' \to \ldots \to \chi_n'' \to \gamma'' \to 0$$

represents the "bottom" of the square, and if in addition β'' is epic, γ'' is monic, and $\beta \to \beta''$ is epic with kernel an epimorphism β' with kernel A', and $\gamma \to \gamma''$ is monic with cokernel γ' a monic with cokernel Z', then one has the new exact sequence Δ''

$$\Delta'': 0 \to A' \to X_1 \to X_2 \oplus Y_1 \oplus X_1'' \to X_3 \oplus Y_2 \oplus X_2'' \oplus Y_1''$$

$$\to X_4 \oplus Y_3 \oplus X_3'' \oplus Y_2'' \to \ldots \to Y_n'' \to Z' \to 0.$$

Moreover, since there are two possible choices of "bottoms" of the square, there is a second sequence of this type available, but it is the same as the one above.

One way of viewing this is as the exact sequence of a certain triple complex. Another is to return to our main Theorem 2.7 and recognize that it is still a true theorem if the abelian category $\mathscr{A}$ is replaced throughout by the category of morphisms $\mathscr{A}^2$. Then, given 2 ses's in $\mathscr{A}^2$

$$\Phi_2: 0 \to a \to b \to c \to 0 \text{ and } \Omega_2: 0 \to x \to y \to z \to 0$$

one obtains the les $S(\Omega_2, \Phi_2)$

$$0 \to \mathrm{Hom}(z,a) \to \mathrm{Hom}(y,b) \to \mathrm{Hom}(x \to y, b \to c) \to \mathrm{Ext}^1(z,a) \to \mathrm{Ext}^1(y,b) \to \ldots$$

where $b \to c$ is thought of as an object of the abelian category $(\mathscr{A}^2)^2$. It is simply a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "b"'] & \bullet \arrow[d, "c"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

in $\mathscr{A}$. If the connecting homomorphism is again denoted by τ_n, then one can also interpret Δ'' as being an element in the class of $\tau^{n+1}\tau^n[\Gamma^n \to \Gamma^{n''}]$, where

$$\tau^n: \mathrm{Ext}^n(\gamma \to \gamma'', \beta \to \beta'') \to \mathrm{Ext}^{n+1}(\gamma', \beta'),$$

$$\tau^{n+1}: \mathrm{Ext}^{n+1}(\gamma', \beta') \to \mathrm{Ext}^{n+2}(Z', A').$$

The process can be pushed through for higher dimensions.

References

[1] H.-B. Brinkmann and D. Puppe, Abelsche und Exakte Kategorien, Korrespondenzen. Lecture Notes in Mathematics 96 (Springer, Berlin, 1969).

[2] H. Cartan and S. Eilenberg, Homological Algebra (Princeton University Press, Princeton, N.J., 1956).

[3] P. Freyd, Abelian Categories (Harper and Row, New York, 1965).
[4] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory (Springer, Berlin, 1967).
[5] B. Mitchell, Theory of Categories (Academic Press, New York, 1965).
[6] I.S. Pressman, Obstructions to liftings in commutative squares. Canadian Journal of Mathematics 23 (1971) 977–982.
[7] I.S. Presman, Functors whose domain is a category of morphisms. Acta Math. 118 (1967) 223–249.
[8] J.-L. Verdier, Categories derivées – mimeographed notes, Institute des Hautes Études Scientifiques.